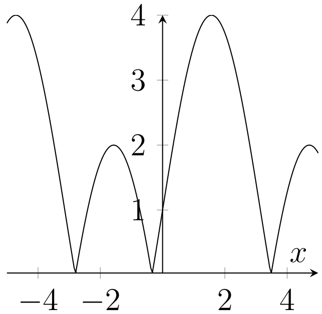
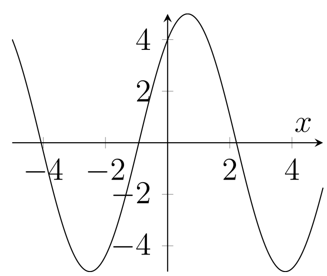
MATH1010 Assignment 1

Suggested Solution

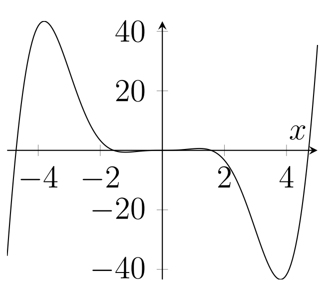
1a.



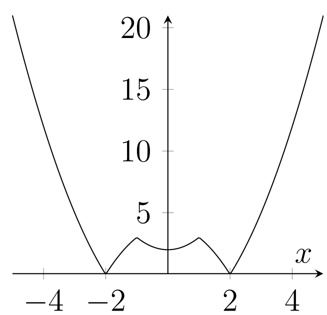
1b.



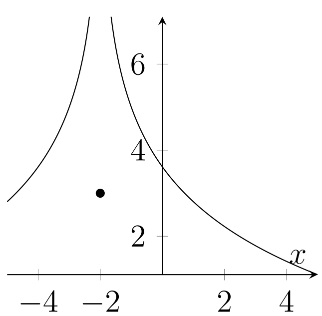
1c.



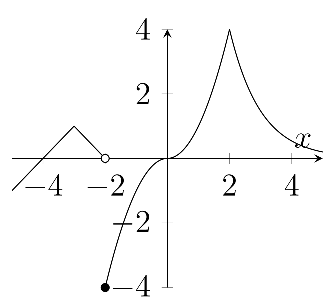
1d.



1e.



1f.



2. (a)

$$\begin{aligned} &\lim_{t\to 0} \frac{\ln(e+t)-1}{et} \\ &= \lim_{t\to 0} \frac{\ln(e+t)-\ln(e)}{t} \cdot \frac{t}{et} \\ &= \frac{1}{e^2} \end{aligned}$$

(b)

$$\begin{aligned} &\lim_{x\to -2} \frac{|x|^3-8}{x^4-16} \\ &= \lim_{x\to -2} \frac{-x^3-8}{x^4-16} \\ &= \lim_{x\to -2} -\frac{x^2-2x+4}{(x-2)(x^2+4)} \\ &= \frac{3}{8} \end{aligned}$$

(c)

$$\begin{aligned} &\frac{\sqrt{x^2-2x+1}}{x-1} \\ &= \frac{|x-1|}{x-1} \\ &= \begin{cases} 1 & x>1 \\ -1 & x<1 \end{cases} \end{aligned}$$

$$\lim_{x\to 1} \frac{\sqrt{x^2-2x+1}}{x-1} \text{ dose not exiest}$$

(d)

$$\frac{|\sin x|}{e^x - e^\pi} = \frac{|\sin x - \sin \pi|}{x - \pi} \cdot \frac{x - \pi}{e^x - e^\pi}$$

$$\begin{aligned}
&\lim_{x \to \pi^+} \frac{|\sin x - \sin \pi|}{x - \pi} \cdot \frac{x - \pi}{e^x - e^\pi} \\
&= \lim_{x \to \pi^+} -\frac{\sin x - \sin \pi}{x - \pi} \cdot \frac{x - \pi}{e^x - e^\pi} \\
&= \frac{1}{e^\pi} \\
&\lim_{x \to \pi^-} \frac{|\sin x - \sin \pi|}{x - \pi} \cdot \frac{x - \pi}{e^x - e^\pi} \\
&= \lim_{x \to \pi^-} \frac{\sin x - \sin \pi}{x - \pi} \cdot \frac{x - \pi}{e^x - e^\pi} \\
&= -\frac{1}{e^\pi}
\end{aligned}$$

$$\lim_{x \to \pi} \frac{|\sin x}{e^x - e^\pi} \text{does no exist}$$

(e)

$$\frac{x}{|x|} \sin \frac{|x|}{x} = \sin 1 \quad \text{for } x \neq 0$$

$$\lim_{x \to 0} \frac{x}{|x|} \sin \frac{|x|}{x} = \sin 1$$

(f)

$$\begin{aligned}
&\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}\\
&=\lim_{x\to 0}\frac{1-\cos^2 x}{1-\cos x}\\
&=\lim_{x\to 0} 1+\cos x\\
&=2
\end{aligned}$$

(g)

$$\begin{aligned}
&\lim_{x\to +\infty}\frac{8e^{2x}-9x^2}{(6e^x+x^4)^2}\\
&=\lim_{x\to +\infty}\frac{8-9\frac{x^2}{e^{2x}}}{36+12\frac{x^4}{e^x}+\frac{x^8}{e^{2x}}}\\
&=\frac{2}{9}
\end{aligned}$$

(h)

$$\begin{aligned}
&-\frac{1}{x}<\frac{\sin x}{x}<\frac{1}{x}\quad \text{for } x>0\\
&\lim_{x\to\infty}\frac{1}{x}=0\\
&\lim_{x\to\infty}-\frac{1}{x}=0\\
&\text{(By Sandwich Theorem)} \lim_{x\to\infty}\frac{\sin x}{x}=0
\end{aligned}$$

(i)

$$\begin{aligned}
&\lim_{x\to-\infty}(x+\sqrt{x^2+4x})\\
&=\lim_{x\to-\infty}\frac{-4x}{x-\sqrt{x^2+4x}}\\
&=\lim_{x\to-\infty}\frac{-4}{1-\frac{\sqrt{x^2+4x}}{x}}\\
&=\lim_{x\to-\infty}\frac{-4}{1-\frac{|x|}{x}\sqrt{1+\frac{4}{x}}}\\
&=-2
\end{aligned}$$

(j)

$$\begin{aligned}
&-\frac{1}{(\ln x)^2+2015}<\frac{\cos x}{(\ln x)^2+2015}<\frac{1}{(\ln x)^2+2015}\quad\text{for } x>0\\
&\lim_{x\to+\infty}-\frac{1}{(\ln x)^2+2015}=0\\
&\lim_{x\to+\infty}\frac{1}{(\ln x)^2+2015}=0\\
&\text{(By Sandwich Theorem) }\lim_{x\to+\infty}\frac{\cos x}{(\ln x)^2+2015}=0
\end{aligned}$$

(k)

$$\begin{aligned}
&\lim_{x\to-\infty}\sqrt{x^2+x+1}-\sqrt{x^2-x+1}\\
&=\lim_{x\to-\infty}\frac{2x}{\sqrt{x^2+x+1}+\sqrt{x^2-x+1}}\\
&=\lim_{x\to-\infty}\frac{2}{-\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}-\sqrt{1-\frac{1}{x}+\frac{1}{x^2}}}\\
&=-1
\end{aligned}$$

3. (a) i. $f'(x)=\beta(x^{\beta-1}-1)$

ii. For $0 < x < 1$, $f'(x) < 0$

f is strictly decreasing on $(0, 1]$

iii. For $x > 1$, $f'(x) > 0$

f is strictly increasing on $[1, +\infty)$

iv. $f'(x) = 0$ if and only if $x = 1$

$f'(x) < 0$ for $0 < x < 1$

$f'(x) > 0$ for $x > 1$

$f(x)$ attain minimum at $x = 1$

(b) Put $x = 1 + r$. For $r > -1$, we have $x > 0$

$$f(1+r) \geqslant f(1) \quad \text{for } r > -1(1+r)^\beta + \beta - 1 - \beta(1+r) \geqslant 0$$

Thus $(1+r)^\beta \geqslant 1 + \beta r$ for any $r \in (-1, +\infty)$

4. (a) x-intercepts:$y = 0, x = \frac{5}{3}$

y-intercepts:$x = 0, y = 5$

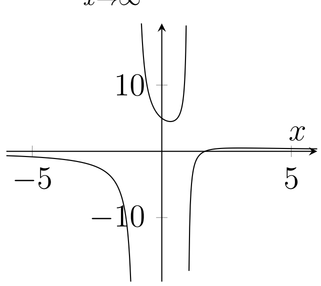
(b) $f'(x) = \frac{-3x^2+10x-3}{(x^2-1)^2}$,$f''(x) = \frac{2(3x^3-15x^2+9x-5}{(x^2-1)^3}$

(c) maximum point:$(3, \frac{1}{2})$

minimum point:$(\frac{1}{3}, \frac{9}{2})$

(d) vertical asymptotes:$x = \pm 1$

Since $\lim_{x\to\infty} f(x) = 0$,horizontal asymptotes:$y = 0$

(e) 

5. Let V be volume of the water, $\dfrac{dV}{dt} = \dfrac{1}{4}$

Let h be depth of the water

$$Then \quad V = \frac{1}{3}\pi r^2 h \quad \text{and} \quad r = \frac{3}{2}h$$

$$V = \frac{3}{4}\pi h^3$$

$$\frac{dV}{dt} = \frac{9}{4}\pi h^2 \frac{dh}{dt}$$

$$\left.\frac{dh}{dt}\right|_{h=1} = \frac{1}{9\pi}$$